Useless Notes #1: Fun with Orbifolds and the Barnes-Wall Lattice.

1 Pure-ish 3D Gravity?

Every known example (that I know) of AdS/CFT has a compact dimension on the same order as the AdS scale. This seems rather annoying. In particular, in AdS_3/CFT_2, this corresponds to the presence of a lot of primaries of weight $O(1)$. Therefore, it seems like a good exercise to attempt to kill as many of the low weight primaries as possible. The potential for making sure that the compact dimension is of order $O(1)$ in size is the work on extremal CFTs, but this seems too hard - one could in principle imagine finding a family of CFTs whose first non-identity primary is of weight $log(c)$: then the size of the compact dimension and whatever stringy effects may be in play would still be rather large, but eventually a scale separation would develop.

One obvious approach to try is orbifolding Narain lattice models by translations and rotations. Unfortunately, this won't get all the way down to the Virasoro algebra, because of the rotationally invariant combinations of the oscillator modes will never be killed. These states do seem to have some sort of nice large-N type behaviour, though. For instance, the partition function can be calculated without too much difficulty: In the chiral case, one has

$$Z(q) = \prod_{i \leq j} \frac{1}{1 - q^{i+j}} \tag{1}$$

and in the non-chiral case

$$Z(q, \overline{q}) = \prod_{i \leq j} \frac{1}{|1 - q^{i+j}|^2} \prod_{i,j} \frac{1}{1 - q^i \overline{q}^j} \tag{2}$$

It's possible to calculate the large N expansion of the correlation functions of all these states and it all seems a bit to nice to not have some interpretation on the gravity side, but since I haven't managed to find any CFT with this as its low-lying spectrum, I'm not going to bother with its analysis too much.

2 The Barnes-Wall lattice and the Clifford Group

The closest that I have managed to get to that optimal low-lying spectrum is using something called the Barnes-Wall lattices. Since I have, in fact, never heard of these things until Noam Elkies told me about them, I will mainly be drawing from the papers math/0207186, math/0403480 and math/0001038 for information about them.

The fastest, although not the most explicit, definition of the Barnes-Wall lattices is to first define the balanced lattice $M_1 \subset \mathbb{R}^2$ consisting of integer linear combinations of $(0,1), (1,0), (1/\sqrt{2}, 1/\sqrt{2})$, and $(1/\sqrt{2}, -1/\sqrt{2})$. Then the balanced Barnes-Wall lattice is $M_k = M_1^{\otimes k} \subset \mathbb{R}^{2^k}$, and the actual Barnes-Wall lattice is $L_k = M_k \cap \mathbb{Q}^{2^k}$. The symmetry group of the balanced Barnes-Wall lattice M_k is the Clifford group $\mathcal{C}_k$ and the symmetry group of L_k is an index two subgroup of $\mathcal{C}_k$. I haven't been able to find a name for this subgroup - call it $\mathcal{D}_k$.

The good behaviour of the Barnes-Wall lattice is that when $k \geq 3$ is odd, L_k is even, unimodular, and has the minimum norm of a non-zero vector equal to $2^{\frac{k-1}{2}}$. (Note that the norm of a vector is the square of its length.) Let $N = 2^k$.

It will be necessary to understand the behaviour of the Clifford group decently well in order to analyze the twist sector, so some analysis will be presented here. An important subgroup of $\mathcal{C}_k$ is the 'extraspecial 2-group' $E(k) \simeq 2_+^{1+2k} \subset O(2^k, \mathbb{R})$ generated by the Pauli matrices $1 \otimes \cdots \otimes \sigma_x \otimes \cdots \otimes 1$ and $1 \otimes \cdots \otimes \sigma_z \otimes \cdots \otimes 1$. Note that σ_y does not appear, only $i\sigma_y$ does. $E(k)$ naturally has a representation, call it $V \simeq \mathbb{R}^k$. Inspection reveals that $E(k) \subset L_k$. Another definition for the Clifford group is as the normalizer of $E(k)$ in $O(2^k, \mathbb{R})$. That is, the Clifford group C_k is the set of orthogonal matrices whose action by conjugation sends every element of $E(k)$ to another. The centralizer of $E(k)$ turns out to just be the two matrices $\pm I^{2^k}$. Therefore, an element of the Clifford group is fixed up to sign by its action on $E(k)$. In turn, note that $E(k)/\pm I^{2^k} \simeq \mathbb{F}_2^{2k}$ since the Pauli matrices commute up to sign. Examining whether each matrix is symmetric or anti-symmetric, which can be determined from withing $E(k)$ by asking whether the square of the element is ± 1, gives a quadratic form $\mathbb{F}_2^{2k}$ and so the outer autormorphism group of $E(k)$ turns out to be the group called $O^+(2k, 2) \subset GL(2k, \mathbb{F}_2)$. As it turns out that every outer automorphism of $E(k) \simeq 2_+^{1+2k}$ can be achieved by an element of C_k, there is another formulation of the Clifford group as the semidirect product $2_+^{1+2k}.O^+(2k, 2)$.

3 The CFT

Since I don't want to think hard about anomalies, I will only be doing the non-chiral case. The chiral case should be similar, provide that the orbifold exists. There is a natural CFT N_k which is the lattice theory consisting of a holomorphic and anti-holomorphic Barnes-Wall lattice CFT. This has an automorphism group containing $Aut(L_k)$, and so one can construct the orbifold CFT BW_k which is the quotient of N_k by $Aut(L_k)$.

BW_k has three types of states, the invariant combinations of the oscillators, the momentum/winding states, and the twist states. By construction, the momentum/winding states all have weight $O(\sqrt{N})$. Note that this is very different from the behaviour of e.g. the symmetric product orbifolds! I will now argue that the twist states have the same behaviour.

3.1 Twist States

Suppose one has some rotation R on V_k. If the eigenvalues of R are $e^{i\theta_i}$, then the minimal possible energy of a state twisted by R is $\sum_i f(\theta_i)$, there f is some function that I haven't bothered to look up, but which is $O(1)$ whenever θ_i itself is $O(1)$. Suppose that $R \in \mathcal{C}_k$. Then, there are three cases: $R = \pm I^{2k}$, $R \in E(k)$ but $R \neq \pm I^{2^k}$, and $R \notin E(k)$. In the first case, R is either the identity, in which case it isn't a twist state at all, or R is the negative of the identity, in which case the weight is $O(N)$. In the next case, either half of the eigenvalues are $+1$ and half are -1, or half of the eigenvalues are $+i$ and the other half $-i$. In either of these cases, the ground state weight is also $O(N)$.

It is this final case which is non-trivial. The action on V itself is rather gnarly, but it is much easier to figure out what is happening in the adjoint actions on the space of matrices $V_k^\vee \otimes V_k \simeq V_k \otimes V_k$. This space is spanned by tensor products of $1, \sigma_x, i\sigma_y$, and σ_z. Since these are exactly the elements of $E(k)$, R must act on the matrices by permuting them according to the non-trivial element of $O^+(2k, 2)$ and by the flips of some signs.

By standard linear algebra reasons, an element of $GL(2k, \mathbb{F}_2)$ cannot fix more than half of the elements of $\mathbb{F}_2^{2k}$, so $O(N^2)$ of them must be in non-trivial cycles. No matter which way the signs flip, if you set some $O(1)$ cutoff, e.g. $|\theta_i| > \pi/2$, then an $O(1)$ fraction of the eigenvalues originating from a cycle of any length will be above that cutoff.

Now suppose that the eigenvalues of the action of R on V are θ_i, and that $k << N$ of them satisfy $|\theta_i| > \pi/4$. Then the eigenvalues of the action of R on $V \otimes V$ is by $e^{\theta_i - \theta_j}$. Therefore, an $O(kN)$ number of eigenvalues of the adjoint action are larger than $\pi/2$ in absolute value, a contradiction. Therefore, an $O(N)$ number of eigenvalues of the action of R on V must be larger than $\pi/4$ and so the weight of the ground state is $O(N)$

Therefore, every twist state has energy $O(N)$.

3.2 Large N and Invariants

There is now a pretty nice division of energies here: From energies of order 1 to $E \sim O(\sqrt{N})$, only the vector-like invariant combinations of the oscillator modes $\partial^n X^\mu$ and $\overline{\partial}^n X^\mu$ contribute. At $E \sim \sqrt{N}$, the first momentum/winding states start appearing, continuing in an increasing plethora until the giant morass of twist states come in at $E \sim N$.

3.2.1 $E \sim 1$

Even when just considering $E(k)$, the representation V is an irreducible real representation. Therefore, the only invariant of $V \otimes V$ is the natural metric. Since $-I^{2k}$ is an element of $E(k) \subset \mathcal{D}_k$, only V^{2m} can have an invariant tensor. One of the cited papers computes the *symmetric* invariants of $\mathcal{C}_k$ and shows that they are constant in number for large enough N and that there are no such nontrivial invariants until $Sym^8 V$, at which point there is one related to the Hamming code.

Since the relevant fields are vector-like instead of adjoint, the 'single-trace' states are just those of the form $\partial^i X^\mu \partial^j X_\mu$, $\partial^i X^\mu \overline{\partial}^j X_\mu$, and $\overline{\partial}^i X^\mu \overline{\partial}^j X_\mu$. A simple direct calculation shows that their normalized three-point function is of order $1/\sqrt{N}$. More on the analysis of the higher energy states later after I get a better grasp on the number of non-symmetric tensor invariants.

3.2.2 $E \sim \sqrt{N}$

I haven't really done any analysis of the regime, but I do get the impression that the momentum modes have a very suspicious resemblance to D-branes. The action of the Clifford group on the first non-trivial winding modes I think is transitive, and has a stablizer which seems to be related to $\mathcal{C}_{k-1}$. Due to the symmetry bringing together multiple bits, the momentum mode seems to be able to do something like absorb some number of loose ∂X indices. Haven't thought much about it, but seems analyzable.

3.2.3 $E \sim N$

Who knows, man. Here be dragons.

4 Codes and Codes Galore

4.1 The Upshot and the Lowdown

The low energy behavior of this theory is given by the theory of invariants of $\mathcal{D}_k$. When the number of legs n is less than or equal to $2k+2$, the invariants of the Clifford group correspond exactly to the even self-dual codes on n bits. Beyond this, 'trace relations' of a sort start arising and it seems like it is also at $O(k)$ where the first invariants of $\mathcal{D}_k$ which are negative in parity under $\mathcal{C}_k$ which start to pop up.

The number of even self-dual codes on n bits, where n is even, is exactly

$$\sum_{i=1}^{\frac{n}{2}-1} \left(2^i + 1\right). \tag{3}$$

This scales as $2^{\frac{1}{2}n^2+O(n)}$ and since the mathematicians say that almost all such codes have trivial automorphism group, the number of symmetric invariants scales the same way. Therefore, the number of (very-)low lying states grows faster than Hagedorn! This is very much not pure 3d gravity, but this isn't the Hagedorn growth of the typical stringy background, either.

At energies much larger than k but much smaller than N, a quick back-of-the-envelope calculation of the number of invariants comes out to something like $(N)^{n-k}$ or thereabouts, which is Hagedorn, so presumably the strings start showing up here.

Meanwhile, there is indeed a very nice factorization of the correlation functions with a $1/N$ expansion, where the 'single-trace' states are exactly the irreducible codes. The good behaviour of the $1/N$ expansion is *so* good, in fact, that it is blithely unaware of the mayhem that starts occuring at order k and continues until the total number of tensor legs starts to compete with N itself. This has got to be the cleanest demonstration I've seen of a phenomenon that Dan Jafferis once told me about where a perturbative canonical quantization of the interior of a black hole actually comes out to too high an entropy, so it must be that the subleading corrections to the inner product must cause a large number of null states to appear, cutting down the size of the Hilbert space. This is precisely what must happen here, and it happens at a point that the peturbation theory seems completely oblivious to!

I am apparently at a loss for words as to how to describe the inexplicably beautiful and deep structure of this theory.

4.2 The Counting of the Invariants

The vector space $V_k \simeq \mathbb{R}^N$ can be represented by linear combinations of generators μ_ν, where $\nu \in \mathbb{F}_2^k$. The Clifford group is generated by the following three types of actions.

1. $\mathrm{diag}\,(-1)^{q(\nu)+a}$, where q ranges over all $0,1$-valued quadratic forms on $\mathbb{F}_2^k$ and $a \in \mathbb{F}_2$. Without loss of generator-ness, one can restrict to $q = e_i e_j$ for $i \leq j$.

2. $AGL(k,2)$, the affine general linear group on $\mathbb{F}_2^k$, sending μ_ν to $\mu_{A\nu+\mu}$.

3. And finally, the matrix $h \otimes I_2 \otimes \ldots \otimes I_2$, where h is the Hadamard matrix, $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

Only the third generator has a factor of $\sqrt{2}$. Therefore, the subgroup $\mathcal{D}_k$ of $\mathcal{C}_k$ is exactly those elements which have an even number of h's. As such, invariants of $\mathcal{D}_k$ can be split into those which are of even and odd parity under h.

Now, to consider $V_k^{\otimes n}$. This vector space is generated by μ_M, where M is a $k \times n$ matrix of bits. This matrix is originally interpreted as n columns of length k, but for most of the rest of the analysis, it is more productive as k rows of length n. Due to the presence of $-I_N$ as an element of the automorphism group, all invariants have n even, so I will be assuming n is even from now on.

The first set of generators leaves each monomial fixed and possibly changes its sign. The quadratic forms just ensure that each pair of rows is orthogonal, by the typical "multiply the entries and add" dot product rule. In particular, rows being orthogonal to themselves imply that they have a even number of elements, and thus have zero dot product with the all ones vector 1^n. Therefore, the span of the rows (adding in 1^n for good measure) forms a self-orthogonal, but not necessarily self-dual, code. (Given a subspace $C \subset \mathbb{F}_2^n$, its orthogonal complement is $C^\perp$ and consists of all vectors which are orthogonal to all of the elements of C. A code is self-orthogonal if $C \subseteq C^\perp$ and self-dual if $C = C^\perp$. It seems that generally self-orthogonal codes are taken to always contain the vector 1^n.)

The action of the second set of generators permutes the μ_M's, acting on each M by some linear recombination of its rows and by potentially adding 1^n to arbitrary rows. Let S_M be the span of the rows of M along with 1^n. Then, $AGL(k,2)$ sends M to M' if and only S_M and $S_{M'}$ agree.

Given a self-orthogonal code C, one can now define $\mu_{C,k} = \sum_{M, S_M = C} \mu_M$ and

$$f_k(C) = \sum_{M, S_M \subseteq C} \mu_M. \tag{4}$$

These are both invariants of the 'parabolic' subgroup generated by the first two types of generators. When the dimension of C exceeds $k+1$, $\mu_{C,k}$ vanishes. It is not too hard to show that $\mu_{C,k}$ for $\dim(C) \leq k+1$ forms a basis of the parabolic invariants and that $f_k(C)$ for $\dim(C) \leq k+1$ does too.

The tensor $f_k(C)$ factorizes as $(f_1(C))^{\otimes k}$ (note the opposite tensor decomposition as initially). Therefore, the Hadamard matrix acts as $(h^{\otimes n} f_1(C)) \otimes (f_1(C))^{\otimes k-1}$. Then,

$$h^{\otimes n} f_1(C) = h^{\otimes n} \sum_{c \in C} \mu_c = 2^{-\frac{n}{2}} \sum_{c' \in \mathbb{F}_2^n} \sum_{c \in C} (-1)^{\langle c, c' \rangle} \mu_{c'} = 2^{-r} \sum_{c' \in C^\perp} \mu_{c'}, \tag{5}$$

where $r = n/2 - \dim C$. Therefore, whenever C is self-dual, $f(C)$ is an invariant of the Clifford group. Some further analysis can show that *every* invariant of the Clifford group can be obtained in this way. However, when $n \geq 2k+4$, these invariants are not linearly independent. There are the additional determinant-like invariants of D_k which are not of this type. More about these later, once I have had the chance to analyze them.

For now, I will focus on these Clifford invariants, and not worry too much about the trace relations or the negative parity invariants.

4.3 The Multiplication of the Codes

While there are other ways to count the number of even self-dual codes which allow easy generalization, here's a particularly direct one.

Suppose you have a self-dual code $C \subset \mathbb{F}_2^n$. There are two possiblities for the last two bits: either all four combinations of two bits will appear, or the last two bits will be always either 00 or 11. (Other possibilites are ruled out by self-duality.)

First consider the latter case, where the last two bits always match. Therefore, $(0 \cdots 011)$ is an element of $C^\perp = C$. This means that C decomposes as a direct sum $C' \oplus \delta$, where δ is the trivial self-dual code on two bits, $\{00, 11\}$.

Now suppose that all four combinations come up. Each codeword in C can be expressed as $(\vec{a} b_1 b_2)$. Let C' be the set of $\vec{a}$ which appear in a codeword where $b_1 = b_2$ and let $\vec{c}$ be such that $(\vec{c}10) \in C$. C' is self-orthogonal, since the b's do not contribute to the inner product and if $\vec{a}$ is orthogonal to C', then either $(\vec{a}00)$ (if $\vec{a} \cdot \vec{c} = 0$) or $(\vec{a}11)$ (otherwise) will be orthogonal to C, so a must be in C', so C' is self-dual as well. Given a self-dual C', then given an odd $\vec{c}$, the b's can be determined by taking a dot product with $\vec{c}$, recovering C. There are 2^{n-2} choices for $\vec{c}$, which is defined up to $2^{n/2-1}$ shifts, so there $w^{n/2-1}$ choices for C given C'.

Repeatedly iterating these two procedures gives the desired formula.

Placing two tensors next to each other, giving $f_k(C) \otimes f_k(C')$, gives f_k on the direct sum code: $f_k(C) \otimes f_k(C') = f_k(C \oplus C')$. Therefore, calculating tensor contractions of codes reduces to contracting pairs of indices in a single code. In turn, this process is completely unaffected by the presence of the k factors, so can examine $f_1(C)$. Some staring at the formula $f_1(C) = \sum_{c \in C} \mu_c$ reveals that if the two bits corresponding to the contraction are always the same in C, then the contraction is $2f_1(C')$, whereas if all four bit combinations appear, one gets $f_1(C')$, where C' is defined as in the construction in the previous paragraph.

When all the indices are contracted, one is left with

$$\left(2^{\dim(C \cap \delta^{\oplus n/2})}\right)^k = N^{\dim(C \cap \delta^{\oplus n/2})}. \tag{6}$$

The inner product of a tensor with itself gives $N^{\dim C} = N^{n/2}$, so the normalization factor is $\sqrt{N}^{-n/2}$.

When evaluating correlation functions, the normal ordering of the operators means that the tensor contractions only run between separate blocks of operators. In particular, the contractions never run from an irreducible self-dual code to itself.

Consider adding in each block one at a time. At a intermediate stage of the process, the n_o free legs will possess a code C_o and the new block will have n_i with the code C_i. Then, n_c of the legs will be contracted together. There is a map $C_o \oplus C_i \to \mathbb{F}_2^{n_c}$ consisting of taking the difference (which equals the sum) of the ends of the contraction legs. This comes out to the number of constraints imposed by contraction, so this new block adds

$$n_i - \dim \operatorname{im}(C_o \oplus C_i \to \mathbb{F}_2^{n_c}). \tag{7}$$

to the dimension of the space of consistent configurations.

One has a chain of inclusions of subspaces of $\mathbb{F}_2^{n_c}$ (stripping out various 0 direct summands):

$$(C_o \oplus C_i) \cap (0 \oplus \delta_i^{n_c} \oplus 0) \subseteq C_o \cap (0 \oplus \mathbb{F}_2^{n_c}) \subseteq \operatorname{im}(C_o \oplus 0 \to \mathbb{F}_2^{n_c}) \subseteq \operatorname{im}(C_o \oplus C_i \to \mathbb{F}_2^{n_c}). \tag{8}$$

It turns out that the first and last spaces are dual to each other, so let $\dim((C_o \oplus C_i) \cap (0 \oplus \delta_i^{n_i} \oplus 0)) = n/2 - r_i$ and let $\dim \operatorname{im}(C_o \oplus C_i \to \mathbb{F}_2^{n_c}) = n/2 + r_i$. ($r_i$ can be a half-integer.)

The number r can only vanish when the first and last sub-space coincide. This implies that $C_o \cap (0 \oplus \mathbb{F}_2^{n_c})$ must have dimension $n_c/2$ and so C_o and C_i must split as direct summands for the contracted and non-contracted parts, and the contracted parts must coincide.

The final normalized tensor contraction evaluates to

$$\sqrt{N}^{-n_{\text{tot}}/2} N^{n_{\text{tot}}/2 - \sum_i r_i} = N^{-\sum_i r_i}. \tag{9}$$

All r_i's can only vanish when all the tensor contractions match the irreducible codes together, giving the desired factorization of the correlation function.

5 The Siren Allure of Tensor Networks

Before heading on to establishing the correct tree level scaling behavior of the tensor networks, I will provide here an extended digression on the possibility of creating a dynamical tensor network system out of this kind of theory. Nothing in this whole discussion is actually specific to the Barnes-Wall orbifold theory, but will apply to any orbifold of a lattice by rotations.

In order to have a chance of getting something that resembles a tensor network, we need a discretized theory. Luckily, the free boson theory is very easy to discretize: just have a series of beads with position x_i and Hamiltonian $\sum_i \frac{1}{2a}\left(p_i^\mu p_{i,\mu} + (x_i^\mu - x_{i-1}^\mu)^2\right)$. Orbifolding the discrete free boson is similarly trivial: attach elements of the orbifold group (the lattice translations and the orbifolded rotations) to the spaces between neighboring sites and then demand that the wavefunction is invariant under gauge symmetry.

The ground state has zero winding and no twist, so yet again it is incredibly easy to describe: gauge fix away the gauge connection, and then

$$\Psi \propto \exp(-\frac{1}{2}\sum_{i,j} M_{ij} x^{i,\mu} x^j_\mu), \tag{10}$$

where M is some symmetric matrix whose DFT is exactly soluble. In particular, $x^T M x$ is invariant under translating every x_i^μ uniformly.

This ground state is a sort of squeezed state. The most natural way to try and get another squeezed state out of this is to glue a large collection of squeezed states together. Then, on a graph with V vertices, E edges, and N external legs, once can make a wave-function as

$$\Psi(x_1, \ldots, x_N) \propto \int \prod_{e\in E} dx_e \prod_{v\in V} \Psi_v(x_1, \ldots, x_{n_v}) \tag{11}$$

Note that at this point, all the copies of the free boson don't interact with each other, so one might as well just consider a single boson.

If one considers translation invariant squeezed states of the form $\exp(-\frac{1}{2}x^T M x)$, the invariance under shifting all the positions and the constraint that the momenta must sum to zero means that the whole problem is equivalent to that of resistor networks, with the matrix M corresponding to the current response to a given voltage. Since in the continuum limit, the ground state comes from a disk of uniform conductance, the resistor network in the discrete case should be some dense, planar, network of resistors. Once the network becomes sufficiently dense, there should be enough degrees of freedom to make the output response function line up.

The Hamiltonian, by definition, acts on the boundary. But it should be possible to make an infinitesimal redefinition of the internal variables so that the internal tensors are left unchanged. Then, when applying this to excited tensors, this should give some reasonably local notion of bulk time evolution. The locality is due to this corresponding to deformation of contours of the stress tensor. Ensuring approximate unitarity of the bulk evolution might be a bit more complicated, though.

It is amusing to note the eerie similarity between the nature of the contractions of the squeezed states and that of the Clifford invariants: the Clifford invariants are almost like fermionic versions of the squeezed states.

The Hamiltonian acting on these tensors produces things that mixes position and momentum, so it is necessary to consider more general squeezed states. It is, however, possible to reduce this entire problem down to classical symplectic geometry. In order to get to phase space, consider the density matrix corresponding to the state. From this can be produced the Wigner quasiprobability distribution by

$$W(x,p) = \frac{1}{\pi^N} \int d^N x_a d^N x_b \Psi^*(x_a)\Psi(x_b)\delta^N(2x - (x_a + x_b))e^{ip\cdot(x_a - x_b)}. \tag{12}$$

The squeezed nature of the state means that

$$W(v) \propto e^{-1/2v^T M v}, \tag{13}$$

where v is the combined phase space coordinate containing both x and p. The condition on M from being a pure state implies that it comes from a Kähler structure on the symplectic space.

Contractions act as a sort of symplectic reduction: adding the constraints that the difference between the positions and sum of the momenta must vanish and then extremizing over the values of the intermediate position and momentum. There is a bit of funny business in this case since there is an invariance with respect to translations which gives a constraint on total momentum. This can be dealt with by a symplectic reduction with respect to this translation.

In the $\exp(-1/2x^T Mx)$ type squeezed state, for real M, there is no mixing between x and p. The quasiprobability distibution comes out to something with a M on x's and M^{-1} on p's. In the case of planar resistor networks, the behavior of the inverse comes from noting that noting that if one considers the difference of voltage between neighboring nodes and the currents along an edge, there is a constraint for each vertex that the sum of the currents into the vertex must vanish and a constraint for each face that the current around the face must vanish. Therefore, rotating each edge gives a dual voltage on the dual graph whose differences is the current and a dual current which is the difference of voltages. The

resistances which result are the inverses of the resistances of each edge of the original graph. This gives the inverse matrix's system.

If one re-introduces the orbifolding into the free boson theory, then the tensors become equipped with the gauge connection and must be gauge-invariant. The gauge-invariant condition in particular restricts the allowed low-energy tensors to something manageable, unlike the un-orbifolded case.

From the above sort of discussion, it seems like that incomplete OPE closed sectors of theories can be represented by tensor networks with restrictions on the allowed tensors - it is only when the tensors are loosened to be anything in the relevant gauge-invariant Hilbert space that the boundary theory has to correspond to an actually local field theory.

Anyways, that's enough about that.

6 Recovering the Tree Level

Naïvely, the expansion of the connected correlation into tree level and sub-leading corrections is broken: the contraction of the Hamming spider with four 'jumpers' evaluates to N^{-1}, instead of the expected $N^{-\frac{3}{2}}$ of a five-point function. This is due to the projector from the self-dual codes basis to the orthogonal complement of the multi-trace states has sub-leading in N corrections. Actually computing these projected invariants is a bit annoying. For example, I'm getting a formula with a cubic in N in the denominator for the Hamming spider. This complicated behavior is probably due to some sort of demand for poles at the places where null states show up. However, to ensure that the tree-level expansion remains well-defined, you only need to go down so many terms: I think up to $O(N^{-(C-1)/2})$ is enough for a C-trace correction.

Note that computing the projector is sensitive to the presence of null states, so the perturbation expansions here probably are sensitive to k, unlike before. I will be taking k to be larger than the number of insertions at play so that the inner product never deviates by more than a $O(1)$ factor from the leading order behaviour.

Consider the contraction of n_c of the legs of two corrected single-trace tensor T_1 with n_1 legs and T_2 with n_2 legs together. Pick a a-trace self-dual code tensor on $n_1 + n_2 - 2n_c$ legs as a probe to contract against the original contraction. If any trace component of the probe contracts solely with either T_1 or T_2, then the result must vanish by the projected-ness of T_1 and T_2. Therefore, each component must contract with both T_1 and T_2. Acting in the order of putting T_1, the components of the probe one-by-one, and then T_2, gives each component of the probe contributing $r \geq \frac{1}{2}$, so the total inner product must be order at most $N^{-a/2}$. Note that this argument can break when n is sufficiently large, since the superexponential number

of states in play can make a large number of sub-leading corrections add up to something significant. This almost certainly doesn't happen until n becomes the same order as k, where there start being trace relation things, but one can take n fixed and k to infinity and throw out this possibility no matter what the correct threshold is.

Anyways, this shows that the a-trace part of the contraction of T_1 and T_2 (in the projected basis) is at most order $N^{-a/2}$. Suppose that one already knows that the connected contraction of w irreducible tensors goes as at most $N^{-(w-2)/2}$ for w smaller than the actual number of tensors we are interested in. Then, take two connected tensors and expand their contraction. In the worst case, the a-trace part of the contraction can break the graph into a pieces. Then, the contribution can go by at most N^{-b}, where

$$b = \frac{a}{2} + \sum_{i=1}^{a} \frac{w_i - 2}{2} = \frac{a}{2} + \frac{w + a - 2}{2} - a = \frac{w-2}{2}. \tag{14}$$

Since we have already handled the two and three-point case, we are done.