Useless Notes #3: How to Čech Your Actions

I need to do explicit, concrete calculations with differential cohomology. Since I keep on forgetting all the signs, I'm writing it down here. The reason for this is the Chern-Simons action. This has the famous formula

$$\int \frac{k}{4\pi}\mathrm{tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right), \tag{1}$$

where A is the connection, an adjoint-valued one-form. The issue is that this requires a trivial G-bundle, which you are not always afforded; generally you can only specify a connection on local patches and then you have to have transition maps linking the patches.

Often, it is possible to bypass this sort of issue. For example, if the gauge group G is simply connected, all bundles on 3-manifolds are topologically trivial and the above formula works. Another approach is the express the 3-manifold as the boundary of a 4-manifold and then take the integral of $\mathrm{tr}\,(F \wedge F)$. But this is sort of unsatisfying because this is not *local.* The Chern-Simons (and WZW, etc.) theory is supposed to be a local QFT, not some system coupled to some sort of topological junk.

So I went and worked out the correct local form for the Chern-Simons action on a general bundle. In each patch, the Lagrangian density 3-form is the same as (1). But now, a the boundary between patches, there is a further boundary term, then at the codimension two boundary between three patches there is another term and at the codimension three boundary between four patches there is yet another term. This last one is the integral over a zero-dimensional boundary of a zero-form. This zero form will turn out to be valued in $U(1)$ rather than R and is the reason for the discretization of the level. No tomfoolery with extending connections is required.

1 The Čech-de Rham Complex

The mathematical machinery which can handle this chain of boundary terms combines de Rham cohomology (which is the usual thing with n-forms and the exterior derivative) an a system for managing patches and boundaries between patches called Čech cohomology. But now, we have a tensor product between two chain complexes containing anti-commuting things, so the dreaded super-signs come up here. This means that we're going to have to take things a bit carefully.

Let's say that we have a d-dimensional manifold M. We can cover M with simpler open subsets, U_i, indexed by $1 \leq i \leq N$. Then, let the intersection of these patches be

$$U_{i_0 \cdots i_n} = U_{i_0} \cap \cdots \cap U_{i_n}. \tag{2}$$

Then we can start playing make-believe and pretend we are working in a different topological space constructed by gluing together together pieces shaped like

$$\Delta^n \times U_{i_0 \cdots i_n}. \tag{3}$$

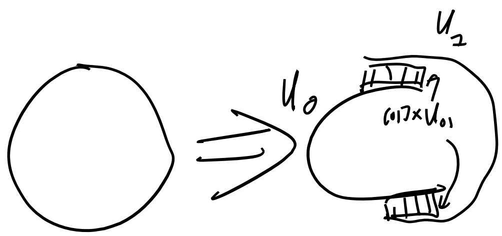

Figure 1: They're the same picture.

Do this gluing right and you get a shape which is just the same as M, cohomologically speaking[1]. Now we can put in the boundary terms as n-forms living inside the interstitial webbing between the patches.

The easiest way to set up all the bookkeeping is to use cohomology's lesser known cousin, cocohomology, which used to be[2] known as homology. Instead of dealing with n-forms, it deals with integration contours and their boundaries.

The usual case of this is to say that a chain is an integer linear combination of fragments of an integration contour. Since it turns out that all integration contours that are relevant here can be built out of simplices, this in practice gives chains as things of the form

$$\sum_{i=1}^{N} n_i [\sigma_i], \tag{4}$$

[1] This is because there is a projection onto M forgetting all the damn simplex nonsense and the fiber of the projection is always a simplex, which is contractible.

[2] And still is.

where N and n_i are some integers and the σ_i are maps $\Delta^n \to M$, where Δ^n is the n-simplex.

If one denotes the n-simplex with vertices $i_0, \ldots, i_n$ as $[i_0 \cdots i_n]$, then the boundary of the n-simplex $[0 \cdots n]$ is

$$\partial[0 \cdots n] = \sum_{k=0}^{n} -^k [0 \cdots \hat{k} \cdots n], \tag{5}$$

where $\hat{k}$ denotes removing k from the sequence and $-^k$ is shorthand for $(-1)^k$. This makes the O.G. chain complex of chains of dimension n with differential given by ∂. Famously, $\partial^2 = 0$ because the boundary of a boundary vanishes. Then the homology groups are the closed chains γ satisfying $\partial\gamma = 0$ (which is why they're called closed) modulo the exact ones of the form $\gamma = \partial\alpha$. But I don't actually care about the homology groups.

Given a n-dimensional chain γ in some manifold M and a n-form ω on M, you can integrate $\int_\gamma \omega$ and we have that the Stokes theorem says $\int_\gamma d\omega = \int_{\partial\gamma} \omega$.

Let's move on to more annoying pastures. In our franken-manifold, we have the natural chains spanned by terms of the form

$$[i_0 \cdots i_n] \times [\sigma], \tag{6}$$

where σ is a map $\Delta^n \to U_{i_0 \cdots i_n}$. And now here come the signs. What we have here is a tensor product of things of mixed anticommutingness and that means we need to make sure the notation is consistent. (See the previous writeup on supersigns.) Now, standard convention dictates that we act by both the homology differential ∂ and the cohomology differential d on the left. But $\langle \partial\gamma, \omega \rangle \neq \langle \gamma, d\omega \rangle$ normally. The least janky resolution to this is to, if R is the super-algebra of scalars, make the cohomology a R-module and homology a R^{op} module, where R^{op} has the order of multiplications reversed. So now in order to bring γ and ω together, the dictums of the notation demand that you write $\langle \gamma^T, \omega \rangle$ to make a scalar that lives in a R-module. Now,

$$\langle (\partial\gamma)^T, \omega \rangle = \langle \gamma^T \partial^T, \omega \rangle = \langle \gamma^T, d\omega \rangle \tag{7}$$

and they lived happily ever after.

Coming back to the chains, we have the test chains $[i_0 \cdots i_n] \times \gamma$ for chains γ in $U_{i_0 \cdots i_n}$ and these collide with co-chains ω. Let

$$\omega_{i_0 \cdots i_n} = ([i_0 \cdots i_n])^T \cdot \omega \tag{8}$$

be a $(d-n)$-form on $U_{i_0 \cdots i_n}$ such that

$$\langle ([i_0 \cdots i_n] \times \gamma)^T, \omega \rangle = \int_\gamma \omega_{i_0 \cdots i_n}. \tag{9}$$

At times I will get lazy and omit all the i's so that $\omega_{i_0\cdots i_n}$ becomes $\omega_{0\cdots n}$ instead[3].

The boundary of the test chain equals

$$\partial([i_0\cdots i_n]\times\gamma) = (\partial[i_0\cdots i_n])\times\gamma + (-)^n[i_0\cdots i_n]\times\partial\gamma. \tag{10}$$

We can split the corresponding differential on ω into $d+\check{\delta}$, where d is the exterior derivative component and δ is the Čech part. One can check that

$$(d\omega)_{i_0\cdots i_n} = -^n d\omega_{i_0\cdots i_n}, \tag{11}$$

$$(\check{\delta}\omega)_{i_0\cdots i_n} = \sum_{k=0}^{n} -^k \omega_{i_0\cdots\widehat{i_k}\cdots i_n}. \tag{12}$$

We can split ω into $\omega^{(0)}+\omega^{(1)}+\cdots+\omega^{(n)}$, where $\omega^{(k)}$ is a $(n-k)$-form on a codimension k boundary. In particular, $\omega^{(n)}$ is a 0-form on a codimension n boundary – if we want our integrals around n-cycles to be valued in $\mathbb{R}/\mathbb{Z}$ instead of $\mathbb{R}$, all we need to do is to make $\omega^{(n)}$ to be $\mathbb{R}/\mathbb{Z}$-valued.

Since all the whatever n-form fluxes twisted by blah blah blah terms have the above sort of property, presumably they ought to be written in this differential cohomology form if you really want to be careful about discretization of the fluxes.

2 True Lie-s

Before proceeding to Chern-Simons action, a quick crash course on the homotopical properties of the Lie groups. One of them is $U(1)$, which is just a circle. A map into $U(1)$ is characterized by the integer elements of the first cohomoology group $H^1(M,\mathbb{Z})$ giving how much the map winds about each cycle. The $U(1)$ bundles are shifted up a degree and are characterized by the integer fluxes valued in $H^2(M,\mathbb{Z})$. This is the boring case, so I will instead focus on the case of a compact, nonabelian Lie group G with simple Lie algebra.

We are interested in the topology of the space of possible G-bundles. This is controlled by the classifying space, BG. This space is a lot like our Čech franken-manifold from earlier, glued together out pieces of manifolds and simplices. In this case, the pieces are shaped like $\Delta^n\times G^n$ with points in G^n parameterized by $g_{01}, g_{12}, \ldots, g_{(n-1)n}$. Then one can define g_{ij} by composing the intermediate group element; think of these as edge elements in a lattice gauge theory with the curvature on the plaquettes forced to vanish. The gluing of the boundaries is then to take the g_{ij}'s in the only reasonable way. There is actually a choice of sign here as to whether g_{ij} is a group element coming from the i side and going to the j side or the

[3] After all, there is no i in team.

other way around. I will take the convention that g_{ij} goes from j to i so that I can write things like $g_{03} = g_{01}g_{12}g_{23}$ without going insane.

More explicitly, this classifying space is constructed by starting with a base point, gluing in $[0,1] \times G$ to give the group elements, then gluing in $\Delta^2 \times G^2$ to encode the composition law, gluing in $\Delta^3 \times G^3$ to do associativity related things, and then gluing in higher things to do their inscrutable higher purposes. A map from M into BG corresponds to the principal G-bundles because such a map can always be deformed to the base point on the codimension zero patches, which then gives a group element in the codimension one boundaries since those land in $[0,1] \times G$. In the codimension two triple intersections, you get the cocycle condition $g_{ij}g_{jk} = g_{ik}$ as needed and everything plays nice to all the higher degrees.

The topology of the space of principal G-bundles over M is controlled by the fundamental groups of BG and the cohomology of M. M can be built by starting with a bunch of points, then gluing in intervals then disks then balls then et cetera. The choice in the possible maps from a n-ball into BG with a given boundary is controlled by $\pi_n(BG)$, which is the group composed of the distict possible maps from the n-ball with boundary the constant map to the base point. When you take into account the constraints and redundancies, you get that this stage is controlled by $H^n(M, \pi_n(BG))$. Note that since G is the loop space of BG by construction, $\pi_n(BG) = \pi_{n-1}(G)$.

I'm taking G to be connected, so $\pi_0(G)$ is trivial. In general, though, G could have a non-trivial (but necessarily finite) π_1. This would mean that there are a finite number of distinct types of G-bundles indexed by $H^2(M, \pi_1(G))$. For simplicity, though, I will take G to be simply connected. The second homotopy group of G vanishes and the third homotopy group is[4] $\pi_3(G) = \mathbb{Z}$. Since for the simply-connected case the first two homotopy groups of G and thus the first three homotopy groups of BG vanish, all G-bundles on a 3-manifold M are deformable to the trivial one. In response to this, I'm going to stick my fingers in my ears, shout LOCALITY LOCALITY LOCALITY, and proceed on as usual. The third homotopy group of G, however, matters. This gives a $\mathbb{Z}$'s worth of non-trivial topological configurations of a G-bundle on a 4-manifold, which is exactly the instanton number, and gives the space of G-bundles on a 3-manifold a non-trivial fundamental group. So the space of bundles looks like a circle. The reason that the Chern-Simons action cannot be made single-valued and the reason for the discretization of the level is that when you go around this circle once, the action shifts by a constant.

The mathematicians have something of a bad habit of specifying groups but not helping you do any calculations because they don't give the generator. So I'm going to describe the

[4]These are derived through some argument involving affine Lie algebras and the loop group. Funnily enough, that same argument gives that *all* homotopy groups except the third of $E6$, $E7$ and $E8$ vanish until the 9th, 11th, 15th, respectively. This might have something to do with M theory.

generator in this case. The way to analyze a Lie group is to pick a maximal commuting subspace of the Lie algebra (this is unique up to conjugation) and then diagonalize *everything* with respect to these mutually commuting operators. The dimension of this Cartan subalgebra is the rank r of the group (which is why $E8$ is called that – it has rank 8). The exponential of the Cartan subalgebra gives the maximal torus in the Lie group – from this you can read off whether it's simply connected, adjoint, or whatever. If you diagonalize the Lie algebra itself with respect to the Cartan subalgebra, you will get a dimension r subspace at zero and a bunch of one-dimensional subspaces. The eigenvalues of each of these subspaces gives the root system of the Lie algebra.

In particular, the subspace associated to each root, the subspace associated to its negative, and the subspace of the Cartan subalgebra coming from the commutator of these, gives a copy of the $SU(2)$ Lie algebra. Correspondingly, to each root there is a map $SU(2) \to G$. Since $SU(2)$ is topologically the same as S^3, this gives an element of $\pi_3(G)$. The generator is then the map coming from any of the shortest roots in the root system (longer ones give covers with degree proportional to the square of their length).

3 The True Chern-Simons Action

A bundle with connection is specified by having on patche i a connection A_i which is a one-form valued in the adjoint representation of the gauge group G and on the boundaries between patches i and j a transition map g_{ij}. These are constrained so that[5] $d + A_j = g_{ji}(d + A_i)g_{ij}$ and $g_{ij}g_{jk} = g_{ik}$. To this, we want to assign a differential cohomology 3-form

$$L = L^{(0)} + L^{(1)} + L^{(2)} + L^{(3)} \tag{13}$$

valued in $\mathbb{R}$ mod something discrete.

I'm going to normalize arbitrarily and set

$$L_0^{(0)} = \text{tr}\left(\frac{1}{2}A_0 \wedge dA_0 + \frac{1}{3}A_0A_0A_0\right) \tag{14}$$

Now, I say trace here but in a general Lie group, there is not a canonical representation for this analagous to the fundamental representation of $SU(N)$. In this case, though, one is only actually using the trace of the product of two Lie algebra elements, which is unique up to rescaling. In particular, the antisymmetry of the one-form A_0 means that $A_0A_0 = \frac{1}{2}\{A_0, A_0\}$ and is in the Lie algebra itself for any representation. Therefore, the choice of representation only really matters for the scaling of the inner product, which I'm going to leave unfixed.

[5]Note that I'm using the mathematician's convention of having the connection being anti-Hermitian.

The condition that we want to demand in order to ensure invariance under gauge transformation and deformation of the boundaries between the patches is that L is closed, so

$$(d+\delta)L = dL^{(0)} + (dL^{(1)} + \delta L^{(0)}) + (dL^{(2)} + \delta L^{(1)}) + (dL^{(3)} + \delta L^{(2)}) + \delta L^{(3)} = 0. \tag{15}$$

On patches, $L^{(0)}$ is already a top-form, so its exterior derivative automatically vanishes. The first non-trivial equation is then

$$dL^{(1)}_{01} = L^{(0)}_1 - L^{(0)}_0 = d\left(\frac{1}{2}\mathrm{tr}\,(A_0 \wedge dg_{01}g_{10})\right) - \frac{1}{6}\mathrm{tr}\,(dg_{01}g_{10}dg_{01}g_{10}dg_{01}g_{10}) \tag{16}$$

We're going to need to look at that last term a bit more closely. One can check that this is a closed 3-form on G. Pick one of the maps $SU(2) \to G$ given by the short roots. This gives something proportional to the corresponding 3-form on $SU(2)$. One can also check that this 3-form is invariant under both the left and right actions and that in the vicinity of the identity it equals

$$-(\mathrm{tr}\,(\sigma_z^2))dxdydz, \tag{17}$$

where one parameterizes the Lie algebra as $ix\sigma_x + iy\sigma_y + iz\sigma_z$. Therefore, if one sets $M = \mathrm{tr}\,((i\sigma_z)^2)$, the integral of the three-form around the three-cycle is $2\pi^2 M$. For the fundamental representation of $SU(N)$, $M = -2$, which means that the action needs to be rescaled by 2π to be valued in $\mathbb{R}/(2\pi\mathbb{Z})$, explaining the $\frac{k}{4\pi}$ in (1).

There is now the integer-integral 3-form

$$\omega^{(1)}_{01} = -\frac{1}{12\pi^2 M}\mathrm{tr}\,\left((dg_{01}g_{10})^3\right). \tag{18}$$

This actually extends to a closed differential cohomology 4-form on BG with

$$\omega = \omega^{(1)} + \omega^{(2)} \tag{19}$$

and

$$\omega^{(2)}_{012} = \frac{1}{4\pi^2 M}\mathrm{tr}\,(dg_{01}dg_{12}g_{20}). \tag{20}$$

Since the $\pi_4(BG) = \mathbb{Z}$ is the first non-vanishing homotopy group of BG, its first cohomology is in degree 4 and equals $\mathbb{Z}$. Thus, there is a integer-valued 4-form (this requires breaking up G into contractible patches) $\tilde{\omega}$ on BG which differs from ω by an exact.

Then, let $\alpha = \alpha^{(1)} + \alpha^{(2)} + \alpha^{(3)}$ and

$$\omega - \tilde{\omega} = (d+\delta)\alpha. \tag{21}$$

Note that α is unique up to exact since the third homology group vanishes in BG as it might well be a point as far as the first three degrees are concerned. If you interpret α as $\mathbb{R}/\mathbb{Z}$

differential cohomology 3-form, we have $(d+\delta)\alpha = \omega$. In particular, we have $d\alpha_{01}^{(1)} = -\omega_{01}^{(1)}$, so this gives the true, local form of the Wess-Zumino term in the WZW model's action.

We now have the next term in the Chern-Simons action:

$$L_{01}^{(1)} = \frac{1}{2}\text{tr}\,(A_0 \wedge dg_{01}g_{10}) - 2\pi^2 M\alpha_{01}^{(1)}. \tag{22}$$

The next constraint is

$$dL_{012}^{(2)} = -L_{12}^{(1)} + L_{02}^{(1)} - L_{12}^{(1)} = -2\pi^2 M d\alpha_{012}^{2} \tag{23}$$

so we have $L^{(2)} = -2\pi^2 M\alpha_{012}^{(2)}$ and $L^{(3)} = -2\pi^2 M\alpha^{(3)}$.

Putting these pieces together, we have the full, local Chern-Simons action:

$$L_{0}^{(0)} = \text{tr}\left(\frac{1}{2}A_0 \wedge dA_0 + \frac{1}{3}A_0 \wedge A_0 \wedge A_0\right), \tag{24}$$

$$L_{01}^{(1)} = \frac{1}{2}\text{tr}\,(A_0 \wedge dg_{01}g_{10}) - 2\pi^2 M\alpha_{01}^{(1)}, \tag{25}$$

$$L_{012}^{(2)} = -2\pi^2 M\alpha_{012}^{(2)}, \tag{26}$$

$$L_{0123}^{(3)} = -2\pi^2 M\alpha_{0123}^{(3)}, \tag{27}$$

where α is $\mathbb{R}/\mathbb{Z}$-valued differential cohomology 3-form on BG satisfying

$$(d+\delta)\alpha = \omega = \omega^{(1)} + \omega^{(2)}, \tag{28}$$

where

$$\omega_{01}^{(1)} = -\frac{1}{12\pi^2 M}\text{tr}\,\left((dg_{01}g_{10})^3\right), \tag{29}$$

$$\omega_{012}^{(2)} = \frac{1}{4\pi^2 M}\text{tr}\,(dg_{01}dg_{12}g_{20})\,. \tag{30}$$